Attainable set of a SIR epidemiological model with constraints on vaccination and treatment stocks

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Abstract

In this paper the controllable spread of some infectious disease is considered. The evolution model of the disease is described by the 3-dimensional nonlinear ordinary differential equations system. Vaccination and treatment are accepted as control parameters of the system. It is assumed that the stocks of vaccination and treatment is limited. Attainable sets of the system are approximately calculated for different control stocks. Graphical results are presented and possible biological applications are discussed.

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1 Introduction

The attainable set of given control system is one of the important notions of the control system theory and is useful tool for investigation of various problems of control systems, differential games and optimization theory (see, e.g. [2], [17], [18], [19], [21]). The attainable set of the control system at given instant of time t consists of points to which the system can be steered at instant of time t. Construction or estimating attainable sets is the one of the fundamental problems of control theory. Predetermining of the attainable sets allows to predict different properties of the control system and to design a control function with desired properties.

The methods developed for approximate calculation of the attainable set of the control system vary with respect to whether or not the function defining the system is linear, or to the constraints which satisfy the control functions. The control functions of the system's being geometric constraint means that the control effort is a kind of quantity limited but not exhausted by consumption, whereas the control functions of the system's being integral constraint is explained in the way that the control resource is limited and is exhausted by consumption. Thus, the control problems concerning the sources that are limited and exhausted by consumption are modeled as control systems with integral constraint on control functions (see, e.g. [3], [17], [23]).

The topological properties and approximate calculation methods of attainable sets of affine control systems (i.e. nonlinear with respect to the phase state vector and affine with respect to the control vector) with integral constraint on the controls have been considered in [8], [9]. These studies are generalized for fully nonlinear case in papers [10], [11], [12] and in [13] an algorithm is presented for approximate calculation of the attainable set. Approximate calculation of attainable sets of controllable physical or biological systems can be used for detailed study of such phenomena in depth.

In this paper we consider evolution of the infectious disease, where vaccination and medicine used for treatment are chosen as control efforts. The mathematical model of this process is described

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by the 3-dimensional system of the nonlinear ordinary differential equations. The mathematical model includes susceptible, infected and recovered groups of individuals which is denoted by S, I and R and therefore is called SIR model. It is also assumed that the stock of vaccine and medicine is limited. The attainable set of given SIR model is calculated.

The paper is organized as follows: In section 2, we present the SIR model to be investigated (Equation 2.1). In section 3, we give the basic conditions which the system have to satisfy (Condions 3.A, 3.B and 3.C) and formulate the theorem from [12] which presents an approximation of the attainable sets of control systems with integral constraint on the controls (Theorem 3.1). In section 4 is shown that the functions appeared in the right hand side of the differential equations described the SIR model satisfy the basic conditions. The theorem characterizing approximation of the attainable sets of the SIR model is formulated (Theorem 4.1). In section 5, the attainable set of the biological system is calculated for various control stocks and graphical results are presented. By using calculated attainable sets, some biological conclusions are discussed.

2 Model formulation

2.1 Preliminary results

There are several mathematical models describing the spread of infectious diseases and these models have been applied for studying of many diseases ([1], [4], [5], [7], [15]) . One of the simplest and basic SIR models is the Kermack-McKendrick model, published in 1927 ([16]). The SIR model is used in epidemiology to compute the amount of susceptible, infected and recovered people in a population. There are also some examples on the use of SIR model for controlling of infectious diseases. These studies are generally related with optimal control theory and their aims are to find optimal control strategies to effect the spread of diseases ([6], [14], [22], [20]).

In this work, we first consider a modified SIR epidemic model ([4]) which is given by the system of differential equations

$$\dot{S}(t) = \nu N(t) - \nu^* S(t) - \frac{\beta I(t) S(t)}{N(t)}, \qquad S(0) = S_0 \geq 0 \tag{2.1a}$$

$$\dot{I}(t) = \frac{\beta I(t) S(t)}{N(t)} - (\gamma + \nu^*) I(t), \qquad I(0) = I_0 \geq 0 \tag{2.1b}$$

$$\dot{R}(t) = \gamma I(t) - \nu^\star R(t), \qquad R(0) = R_0 \geq 0 \tag{2.1c}$$

where $S(t)$ represents the number of susceptible individuals, $I(t)$ represents the number of individuals who are infected, and R(t) represents the number of the individuals who are recovered from the disease with permanent immunity to reinfection. $N(t)$ is the total population i.e., $N(t) = S(t) + I(t) + R(t)$. Here β is a transmission coefficient, γ is recover rate and the constants ν and ν^* stands for death and birth rates respectively. We consider ν, ν^*, β and γ are all positive constants.

Moreover, adding the equations 2.1a, 2.1b and 2.1c, it is derived $\dot{N}(t) = (v - v^*)N$. While $\nu \neq \nu^*$ it is obvious that population is not constant. Because of that its more suitable to consider the proportions of individuals in three epidemiological classes, namely: $s(t) = S(t)/N(t)$, $i(t) = I(t)/N(t)$, $r(t) = R(t)/N(t)$.

Since,

$$\dot{s} = \frac{\dot{S}N - \dot{N}S}{N^2} = \frac{(\nu N - \nu^* S - \beta IS/N)N - (\nu - \nu^*)NS}{N^2} = v - vs - \beta is,$$
$$\dot{i} = \frac{\dot{I}N - \dot{N}I}{N^2} = \frac{[\beta IS/N - (\gamma + \nu^*)I]\,N - (\nu - \nu^*)NI}{N^2} = \beta is - \gamma i - \nu i,$$
$$\dot{r} = \frac{\dot{R}N - \dot{N}R}{N^2} = \frac{[\gamma I - \nu^* R]\,N - (\nu - \nu^*)NR}{N^2} = \gamma i - \nu r,$$

we obtain a new system by fractions,

$$\begin{aligned} \dot{s}(t) &= \nu - \nu s - \beta is, \quad & s(0) &= s_0 \geq 0 \\ \dot{i}(t) &= \beta is - \gamma i - \nu i, \quad & i(0) &= i_0 \geq 0 \\ \dot{r}(t) &= \gamma i - \nu r, & r(0) &= r_0 \geq 0 \end{aligned} \tag{2.2}$$

with subject to the restriction $s + i + r = 1$.

It notices that the compartment $r = r(t)$ does not appear in the first two equation of (2.2). Therefore, we can consider the sub-system

$$\begin{aligned} \dot{s}(t) &= v - vs - \beta is, \quad & s(0) &= s_0 \geq 0 \\ \dot{i}(t) &= \beta is - \gamma i - \nu i. \quad & i(0) &= i_0 \geq 0 \end{aligned} \tag{2.3}$$

and determine r from $r = 1 - s - i$ or from the last equation of the system (2.2).

The feasible region of (2.3) is

$$\Omega = \left\{ x = (s, i) \in \mathbb{R}^2_+ | 0 \leq s + i \leq 1 \right\}, \tag{2.4}$$

which can be verified positively invariant(i.e. for given initial point $x \in \mathbb{R}^2_+$, the trajectory lies in Ω). Hence, the system is both mathematically and epidemiologially well-posed. Thus, we can restrict our attention to the region Ω.

2.2 Model with vaccination and treatment

At this point we modify our model (2.2) by imposing vaccination and treatment.It is assumed that there are two available exterior effort to control the spread of disease: vaccination of the susceptible individuals and treatment of the infected ones. In the model

$$\begin{aligned} \dot{s}(t) &= v - vs - \beta is - su_1, \quad & s(0) &= s_0 \geq 0 \\ \dot{i}(t) &= \beta is - \gamma i - \nu i - iu_2, \quad & i(0) &= i_0 \geq 0 \\ \dot{r}(t) &= \gamma i - \nu r + su_1 + iu_2, \quad & r(0) &= r_0 \geq 0 \end{aligned} \tag{2.5}$$

this actions are modeled by the two dimensional control function $u(\cdot) = (u_1(\cdot), u_2(\cdot)) : [0, 1] \to \mathbb{R}^2$. Here, $u_1(t)$ is the proportion of the consumed vaccines for susceptible individuals, $u_2(t)$ is the proportion of the consumed medicines for treatment of the infected individuals at the instant of time t such that $0 \leq u_1(t) \leq 1$ and $0 \leq u_2(t) \leq 1$. Also, we assume that the control function $u(\cdot) : [0, 1] \to \mathbb{R}^2$ satisfies the inequality

$$\int_0^1 \|u(t)\|^2 \, dt = \int_0^1 \left({u_1}^2(t) + {u_2}^2(t) \right) \, dt \leq \mu_0^2. \tag{2.6}$$

This means that the whole stock of vaccines and medicines which can be consumed to effect the spread of disease, is μ_0.

If we focus on reduced system of (2.5), and for own convenience if we use s=S, i=I, r=R, the system becomes

$$\begin{aligned} \dot{S}(t) &= \nu - \nu S - \beta IS - Su_1, \quad S(0) = S_0 \geq 0 \\ \dot{I}(t) &= \beta IS - \gamma I - \nu I - Iu_2, \quad I(0) = I_0 \geq 0. \end{aligned} \tag{2.7}$$

Here, we determine $R = 1 - S - I$.

3 Approximate calculation of attainable set

In this section, firstly, we give preliminaries for the attainable sets of a general control system. After this, we mention the approximate calculation method for the attainable sets. Detailed information and algorithm about this method is given in [13].

Consider the control system the behavior of which is described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the phase state vector of the system, $u \in \mathbb{R}^m$ is the control vector and $t \in [0, 1]$ is the time.

Let $p > 1$ and $\mu_0 > 0$. Every function $u(\cdot) \in L_p([0, 1]; \mathbb{R}^m)$ such that

$$\left(\int_0^1 \|u(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \mu_0 \tag{3.2}$$

is said to be an admissible control function, where $L_p([0, 1]; \mathbb{R}^m)$ denotes the space of Lebesgue measurable functions $u(\cdot) : [0, 1] \to \mathbb{R}^m$ such that $\|u(\cdot)\|_p < +\infty$, $\|u(\cdot)\|_p = \left(\int_0^1 \|u(t)\|^p \, dt \right)^{\frac{1}{p}}$.

The set of all admissible control functions is denoted my U_p, i.e.

$$U_p = \left\{ u(\cdot) \in L_p\big([0, 1]; \mathbb{R}^m\big) : \|u(\cdot)\|_p \leq \mu_0 \right\}.$$

It is obvious that U_p is the closed ball centered at the origin with radius μ_0 in $L_p\big([0, 1]; \mathbb{R}^m\big)$.

We assume that the right hand side of the system (3.1) satisfies the following conditions:

3.A. The function $f(\cdot) : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous;

3.B. For any bounded set $D \subset [0, 1] \times \mathbb{R}^n$ there exist constants $L_1 = L_1(D) > 0$, $L_2 = L_2(D) > 0$ and $L_3 = L_3(D) > 0$ such that

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \leq [L_1 + L_2(\|u_1\| + \|u_2\|)] \, \|x_1 - x_2\| + L_3 \, \|u_1 - u_2\|$$

for any $(t, x_1) \in D$, $(t, x_2) \in D$, $u_1 \in \mathbb{R}^m$ and $u_2 \in \mathbb{R}^m$;

3.C. There exists a constant $c > 0$ such that

$$\|f(t,x,u)\| \le c(1+\|x\|)(1+\|u\|)$$

for every $(t,x,u) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$.

Let $u_*(\cdot) \in U_p$. The absolutely continuous function $x_*(\cdot) : [0,1] \to \mathbb{R}^n$ which satisfies the equation $\dot{x}_*(t) = f(t, x_*(t), u_*(t))$ a.e. in $[0,1]$ and the initial condition $x_*(0) = x_0 \in \mathbb{R}^n$ is said to be a trajectory of the system (3.1), generated by the admissible control function $u_*(\cdot)$. We denote a trajectory of the system (3.1), generated by the admissible control function $u(\cdot)$ by the symbol $x(\cdot; 0, x_0, u(\cdot))$.

For given $t \in [0,1]$ we set

$$X_p(t;0,x_0) = \{x(t;0,x_0,u(\cdot)) : u(\cdot) \in U_p\}.$$

The set $X_p(t;0,x_0)$ is called the attainable set of the system (3.1) with constraint (3.2) at the instant of time t. It is clear that the set $X_p(t;0,x_0)$ consist of all $x \in \mathbb{R}^n$ to which the system (3.1) is steered at the instant of time $t \in [0,1]$.

The Hausdorff distance between the sets $A \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is denoted by $h(A,E)$ and is defined as

$$h(A,E) = \max\left\{\sup_{x\in A} d(x,E), \sup_{y\in E} d(y,A)\right\},$$

where $d(x,E) = \inf\{\|x-y\| : y \in E\}$.

For given $\sigma > 0$, let

$$S_\sigma = \{s_0, s_1, s_2, \dots, s_K\}$$

be a finite σ-net of unit sphere $S = \{u \in \mathbb{R}^m : \|u\| = 1\}$.

Let $\Gamma = \{0 = t_0 < t_1 < \ldots < t_N = 1\}$ be a uniform partition of the interval $[0,1]$, $\Delta = t_{i+1} - t_i$, $i = 0, 1, \ldots N-1$, $\Gamma^* = \{0 = y_0 < y_1 < \ldots < y_a = H\}$ be a uniform partition of the segment $[0,H]$ and $\Delta_* = y_{j+1} - y_j$, $j = 0, 1, \ldots a-1$.

Setting

$$U^H_{p,\Delta,\Delta_*,\sigma} = \Big\{u(\cdot) \in L_p([0,1];\mathbb{R}^m) \quad : \quad u(t) = y_{j_i} s_{l_i},\ t \in [t_i, t_{i+1}),\ y_{j_i} \in \Gamma^*,\ s_{l_i} \in S_\sigma,$$
$$i = 0, 1, \ldots, N-1 \text{ and } \Delta \cdot \sum_{i=0}^{N-1} y_{j_i}^p \le \mu_0^p \Big\}$$

we define a new control functions set. It is clear that $U^H_{p,\Delta,\Delta_*,\sigma} \subset U_p$.

Since $\Gamma^* = \{0 = y_0 < y_1 < \ldots < y_a = H\}$ is the uniform partition of the segment $[0,H]$ and the diameter of Γ^* is Δ_*, then $y_{j_i} \in \Gamma^*$ can be represented as

$$y_{j_i} = j_i \Delta_*, \tag{3.3}$$

where $0 \leq j_i \leq a$ is an integer. Since the the numbers $y_{j_i} \in \Gamma_*$, $i = 0, 1, \ldots, N-1$, satisfy the inequality $\Delta \cdot \sum_{i=0}^{N-1} y_{j_i}^p \leq \mu_0^p$, then the integers $0 \leq j_i \leq a$, $i = 0, 1, \ldots, N-1$, have to satisfy the inequality

$$\sum_{i=0}^{N-1} (j_i)^p \leq \frac{\mu_0^p}{\Delta \left(\Delta_*\right)^p} \ . \tag{3.4}$$

Taking into consideration (3.3) and (3.6), we can redefine the set $U^H_{p,\Delta,\Delta_*,\sigma}(1;0,x_0)$ as

$$U^H_{p,\Delta,\Delta_*,\sigma} = \Big\{ u(\cdot) \in L_p([0,1];\mathbb{R}^m) \quad : \quad u(t) = \Delta_* j_i s_{l_i},\ t \in [t_i, t_{i+1}),\ 0 \leq j_i \leq a,\ s_{l_i} \in S_\sigma,$$
$$i = 0, 1, \ldots, N-1 \text{ and } \sum_{i=0}^{N-1} j_i^p \leq \frac{\mu_0^p}{\Delta \left(\Delta_*\right)^p} \Big\}.$$

By $Z^H_{p,\Delta,\Delta_*,\sigma}(1;0,x_0)$, we denote the set of all points $z(1) = z(t_N)$ calculated by the recurrent formula

$$z(t_{i+1}) = z(t_i) + (t_{i+1} - t_i)\, f\left(t_i, z(t_i), \Delta_* j_i s_{l_i}\right), \quad z(t_0) = x_0, \quad i = 0, 1, \ldots, N-1, \tag{3.5}$$

where $s_{l_i} \in S_\sigma$ and the integers $0 \leq j_i \leq a$, $i = 0, 1, \ldots, N-1$, satisfy the inequality

$$\sum_{i=0}^{N-1} j_i^p \leq \frac{\mu_0^p}{\Delta \left(\Delta_*\right)^p} \tag{3.6}$$

Following theorem characterizes the Hausdorff distance between the attainable set of the system (3.1) with constraint (3.2) and the set $Z^H_{p,\Delta,\Delta_*,\sigma}(1;0,x_0)$ consisting of finite number of points.

Theorem 3.1. [12] For each given $\varepsilon > 0$ there exists $H(\varepsilon) \in (0,\infty)$, $\Delta^*(\varepsilon) > 0$, $\Delta_*(\varepsilon) > 0$ and $\sigma(\varepsilon) > 0$ such that the inequality

$$h\left(X_p(1;0,x_0), Z^{H(\varepsilon)}_{p,\Delta,\Delta_*(\varepsilon),\sigma(\varepsilon)}(1;0,x_0)\right) < \varepsilon \tag{3.7}$$

holds for every $\Delta \leq \Delta^*(\varepsilon)$.

Remark 3.2. For given $\varepsilon > 0$, the determination of the numbers $H(\varepsilon) \in (0,\infty)$, $\Delta^*(\varepsilon) > 0$, $\Delta_*(\varepsilon) > 0$ and $\sigma(\varepsilon) > 0$ in theorem 3.1 can be found in [12].

Using Theorem 3.1, it is possible to construct an algorithm for approximate calculation of the attainable set of the system (3.1) with constraint (3.2).

For given $\varepsilon > 0$, after the numbers $H(\varepsilon) \in (0,\infty)$, $\Delta^*(\varepsilon) > 0$, $\Delta_*(\varepsilon) > 0$, $\sigma(\varepsilon) > 0$ having been chosen in accordance with the inequality (3.7), the approximate calculation of the attainable set $X_p(1;0,x_0)$ can be reduced into the calculation of a simpler set $Z^H_{p,\Delta,\Delta_*,\sigma}(1;0,x_0)$ consisting of finite number of points $z(1) = z(t_N)$ calculated by the recurrent formula (3.5).

Now let us describe the steps of algorithm to calculate the set $Z^H_{p,\Delta,\Delta_*,\sigma}(1;0,x_0)$.

1. First, for given $\sigma > 0$ we construct the set

$$S_\sigma = \{s_0, s_1, s_2, \ldots, s_K\}$$

which is finite σ-net of the unit sphere $S = \{u \in \mathbb{R}^m : \|u\| = 1\}$ (an algorithm, determining a σ-net on unit sphere S can be found in [13]).

2. In calculation of the set $Z^H_{p,\Delta,\Delta_*,\sigma}(1; 0, x_0)$, the integers $j_0, j_1, \ldots, j_{N-1}$ satisfying the inequality (3.6) are chosen, where $0 \le j_i \le a$, $i = 0, 1, \ldots, N-1$. After choosing the integers $j_0, j_1, \ldots, j_{N-1}$ and elements $\{s_{l_0}, s_{l_1}, \ldots, s_{l_N}\}$ from S_σ , the points of the set $Z^H_{p,\Delta,\Delta_*,\sigma}(1; 0, x_0)$ are calculated by formula (3.5) (detailed algorithm is given in [13]).

4 Calculation of the attainable set of SIR model

Consider the reduced SIR model, the behavior of which is described by the system of equations (2.7). Its easy to show that the right hand side of the system (2.7) satisfies the conditions 3.A, 3.B and 3.C. and for all $t \in [0, 1]$ solutions stay in the set Ω.

Denote

$$\tilde{U}_2 = \left\{u(\cdot) \in L_2\left([0,1]; \mathbb{R}^2\right) : \|u(\cdot)\|_2 \le \mu_0\right\}.$$

So, the set of control functions $\tilde{U}_2$ consists of Lebesgue measurable functions $u(\cdot) : [0, 1] \to \mathbb{R}^2$ such that the inequality (2.6) is satisfied. The set of trajectories of the system (2.1) generated by the control function $u_*(\cdot) \in \tilde{U}_2$ and satisfying the initial condition $(S(0), I(0)) = (S_0, I_0)$ is denoted by symbol $(S(\cdot; 0, S_0, u_*(\cdot)), I(\cdot; 0, I_0, u_*(\cdot)))$.

Let

$$\tilde{X}_2(t; 0, (S_0, I_0)) = \left\{(S(t; 0, S_0, u(\cdot)), I(t; 0, I_0, u(\cdot))) : u(\cdot) \in \tilde{U}_2\right\}.$$

Thus, the set $\tilde{X}_2(t; 0, (S_0, I_0))$ is attainable set of the system (2.7) at instant of time t from initial position $(0, (S_0, I_0))$, where the control functions satisfy the inequality (2.6).

In $\mathbb{R}^2$ the σ-net S_σ can be defined as

$$S_\sigma = \{(\sin i\theta, \cos i\theta) : i = 0, 1, \ldots, r\}, \tag{4.1}$$

where

$$\theta \le \frac{\sigma^2}{2}, \quad r = \left[\left|\frac{2\pi}{\theta}\right|\right], \tag{4.2}$$

$[|\cdot|]$ means the integer part. Since $\theta > 0$, then we have

$$\begin{aligned} & \|(\sin(i+1)\theta, \cos(i+1)\theta) - (\sin i\theta, \cos i\theta)\| = \sqrt{(\sin(i+1)\theta - \sin i\theta)^2 + (\cos(i+1)\theta - \cos i\theta)^2} \\ = \quad & \sqrt{2 - 2\left[\cos(i+1)\theta \cdot \cos i\theta + \sin(i+1)\theta \cdot \sin i\theta\right]} = \sqrt{2(1 - \cos\theta)} \le \sqrt{2\theta} \le \sigma, \end{aligned}$$

and hence the set S_σ defined by (4.1) is a σ-net in $S = \left\{u = (u_1, u_2) \in \mathbb{R}^2 : \|u\| = 1\right\}$, where θ and r satisfy (4.2).

By $\tilde{Z}^{H}_{p,\Delta,\Delta_*,\sigma}(1;0,(S_0,I_0))$, we denote the set of all points $(S(1),I(1))=(S(t_N),I(t_N)))$ calculated by the recurrent formula

$$\begin{cases} \dot{S}(t_{i+1}) &= S(t_i)+\Delta\left[\nu-\nu S(t_i)-\beta I(t_i)S(t_i)-S(t_i)\Delta_* j_i|\sin l_i\theta|\right]\ , & S(0)=S_0\ , \\ \dot{I}(t_{i+1}) &= I(t_i)+\Delta\left[\beta I(t_i)S(t_i)-\gamma I(t_i)-\nu I(t_i)-I(t_i)\Delta_* j_i\,|\cos l_i\theta|\right]\ , & I(0)=I_0\ , \end{cases}$$

where $0\le l_i\le r$, $0\le j_i\le a$ for every $i=0,1,\ldots,N-1$, the integers j_i, $i=0,1,\ldots,N-1$, satisfy the inequality (3.6).

According to the theorem 3.1 we have the validity of the following theorem.

Theorem 4.1. For each given $\varepsilon>0$ there exists $H(\varepsilon)\in(0,\infty)$, $\Delta^*(\varepsilon)>0$, $\Delta_*(\varepsilon)>0$ and $\sigma(\varepsilon)>0$ such that the inequality

$$h\left(\tilde{X}_p(1;0,(S_0,I_0)),\tilde{Z}^{H(\varepsilon)}_{p,\Delta,\Delta_*(\varepsilon),\sigma(\varepsilon)}(1;0,(S_0,I_0))\right)<\varepsilon$$

holds for every $\Delta\le\Delta^*(\varepsilon)$.

5 Graphical representations and conclusions

Here, we provide some numerical simulations of the epidemiological model which describes the theoretical results and predict the evolution of infectious diseases in the population. The model we present here is very general as it can suit any model of diseases like H1N1(Influenza), measles, chicken pox, mumps, etc.

Using the algorithm given in [13], we calculate the set $\tilde{Z}^{H(\varepsilon)}_{p,\Delta,\Delta_*(\varepsilon),\sigma(\varepsilon)}(1;0,(S_0,I_0))$ which is an approximation of the attainable set $\tilde{X}_p(1;0,(S_0,I_0))$ of the system (2.7) at instant of time $t=1$. In these calculations,it is accepted that $p=2$, i.e. the admissible control functions are chosen from the space $L_2\left([0,1];\mathbb{R}^2\right)$. The approximate calculation of the set $\tilde{X}_2(1;0,(S_0,I_0))$ is carried out for different values of control stock parameters μ_0. As we mentioned before, the fraction of recovered individuals can be calculated by $R(t)=1-S(t)-I(t)$.

Note that if $u_1(t)=0$, $u_2(t)=0$, then we obtain system (2.3). We demonstrate a disease scenario with hard situations by choosing parameters as $\nu=0.02$, $\beta=0.75$, $\gamma=0.001$ and for initial fractions of individuals $(S_0,I_0,R_0)=(0.7,0.3,0)$. It is seen in Fig. 1 that if there will not be any control effort, then the situation concerning to the spread of the infection is not favorable and the system needs a control to obtain a profitable result. The fraction of infected individuals in the population is getting fastly increased because of high contact rate(β) and almost all individuals are becoming infected in a short time period. It is also observed that because of low natural recovery rate(γ), the increment in the fraction of recovered individuals is so little .

In this circumstances the spreading of infectious disease must be controlled. In below presented figures (Figure 2, Figure 3, Figure 4) the approximate calculated attainable sets of the system (2.7) are given for different values of control stocks μ_0. Using this calculations, let us discuss the state of the system at the instant of time $t=1$.

If $\mu_0=0.1$, $(S_0,I_0)=(0.7,0.3)$, then according to the Fig.2 we conclude that by using this control stock the fraction of infected individuals in the population can be kept around of 40-45 percent. But it can be also inferred that the proportion of recovered individuals is nearly 5-10 percent. This control stock is not enough to get a favorable result. Infection rate is still high and people who gain permanent immunity against to infection is inadequate.

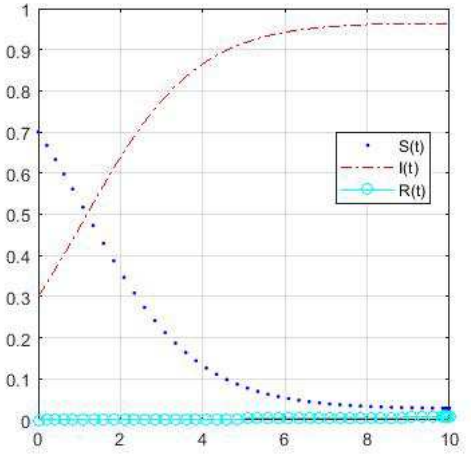

FIGURE 1. Susceptible, infected and recovered fraction without control effect

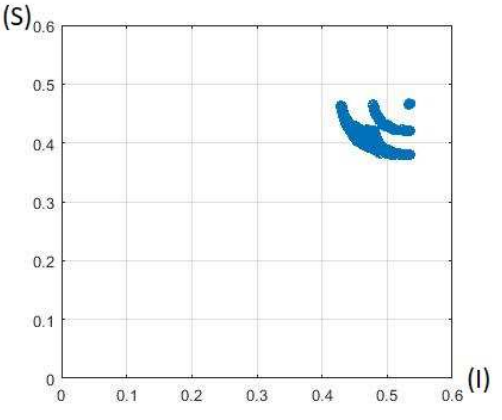

FIGURE 2. Control Stock $\mu_0 = 0.1$

If $\mu_0 = 0.5$, $(S_0, I_0) = (0.7, 0.3)$, then it is observed in Fig. 3 that we obtain better result against to spread of disease. It is achieved to regress the fraction of infected individuals through 25 percent and also the fraction of recovered individuals can be steered to 30 percent.

If $\mu_0 = 1$, $(S_0, I_0) = (0.7, 0.3)$ then according to the Fig.4 we deduce that under such control effort, the fraction of infected individuals can be kept close to zero, i.e. there will be not infected person. It is also understood in the figure, with the effect of vaccination and treatment, the proportion of recovered individuals in the population is able to attained 80 percent.

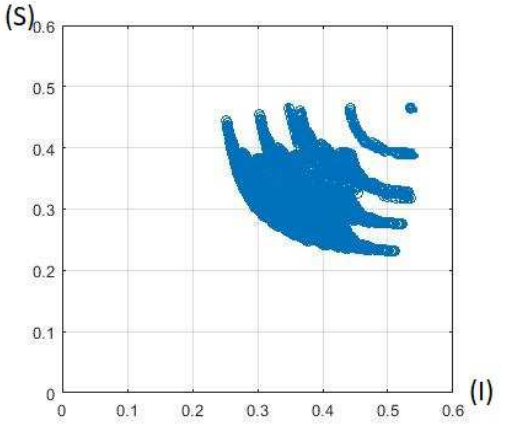

FIGURE 3. Control Stock $\mu_0 = 0.5$

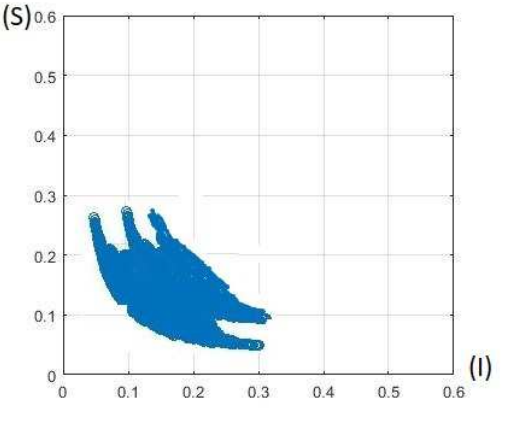

FIGURE 4. Control Stock $\mu_0 = 1$

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